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Criterion for polynomial solutions to a class of linear differential equations of second order

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Abstract

We consider the differential equations $y'' = \lambda_0(x)y' + s_0(x)y$, where $\lambda_0(x)$, $s_0(x)$ are C^∞-functions. We prove (i) if the differential equation has a polynomial solution of degree $n > 0$, then $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$, where $\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0 \lambda_{n-1}$ and $s_n = s'_{n-1} + s_0 \lambda_{k-1}$, $n = 1, 2, \ldots$. Conversely (ii) if $\lambda_n \lambda_{n-1} \neq 0$ and $\delta_n = 0$, then the differential equation has a polynomial solution of degree at most n. We show that the classical differential equations of Laguerre, Hermite, Legendre, Jacobi, Chebyshev (first and second kinds), Gegenbauer and the Hypergeometric type, etc obey this criterion. Further, we find the polynomial solutions for the generalized Hermite, Laguerre, Legendre and Chebyshev differential equations.

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1. Introduction

The question as to whether a second-order linear homogeneous differential equation has a polynomial solution has attracted much interest since the early classification of Bochner of orthogonal polynomials [1]. In 1929, Bochner posed a problem of determining all families of orthogonal polynomials that are solutions of the differential equation

$$(ax^2 + bx + c)y''_n(x) + (dx + e)y'_n(x) - \mu_n y_n(x) = 0. \quad (1)$$

Bochner found that, up to a linear change of variable, only the classical polynomials of Jacobi, Laguerre and Hermite and the Bessel polynomials satisfied a second-order differential equation [2]–[9] of the form (1). In general, the question as to which second-order linear homogeneous differential equation has polynomial solutions (not necessary a sequence of orthogonal polynomials) is not easily answered, since it would involve studying a wide variety of equations, including those with regular and irregular singular points. In this paper, we present a simple criterion for the existence of polynomial solutions of a differential equation of the form

$$y'' = \lambda_0 y' + s_0 y \tag{2}$$

where λ_0, s_0 are C^∞-functions. A key feature of the present work is to note the invariant structure of the right-hand side of (2) under further differentiation. Indeed, if we differentiate (2) with respect to x, we find that

$$y''' = \lambda_1 y' + s_1 y \tag{3}$$

where $\lambda_1 = \lambda_0' + s_0 + \lambda_0^2$ and $s_1 = s_0' + s_0\lambda_0$. If we find the second derivative of equation (2), we obtain

$$y^{(4)} = \lambda_2 y' + s_2 y \tag{4}$$

where $\lambda_2 = \lambda_1' + s_1 + \lambda_0\lambda_1$ and $s_2 = s_1' + s_0\lambda_1$. Thus, for $(n+1)$th and $(n+2)$th derivatives, $n = 1, 2, \ldots,$ we have

$$y^{(n+1)} = \lambda_{n-1} y' + s_{n-1} y \tag{5}$$

and

$$y^{(n+2)} = \lambda_n y' + s_n y \tag{6}$$

respectively, where

$$\lambda_n = \lambda_{n-1}' + s_{n-1} + \lambda_0\lambda_{n-1} \qquad \text{and} \qquad s_n = s_{n-1}' + s_0\lambda_{n-1}. \tag{7}$$

From (5) and (6) we have

$$\lambda_n y^{(n+1)} - \lambda_{n-1} y^{(n+2)} = \delta_n y \qquad \text{where} \quad \delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n. \tag{8}$$

In an earlier paper [11] we proved the principal theorem of the asymptotic iteration method (AIM), namely

Theorem 1. *Given λ_0 and s_0 in $C^\infty(a, b)$, the differential equation (2) has the general solution*

$$y(x) = \exp\left(-\int^x \alpha(t)\,\mathrm{d}t\right)\left[C_2 + C_1\int^x \exp\left(\int^t (\lambda_0(\tau) + 2\alpha(\tau))\,\mathrm{d}\tau\right)\mathrm{d}t\right] \tag{9}$$

if for some $n > 0$

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} \equiv \alpha. \tag{10}$$

The present paper is not about a classification of orthogonal polynomials which is a well-established problem in the literature [1–8]. Rather, the principal goal of the present paper is to characterize when equation (2) has a polynomial solution. In the next section, we shall show that the differential equation (2) has a polynomial solution of degree n, if for some $n > 0, \delta_n = 0$. In section 3, we show through a detailed analysis that the classical differential equations of Laguerre, Hermite, Legendre, Jacobi, Chebyshev (first and second kinds), Gegenbauer and the Hypergeometric type, etc obey this criterion. In section 4, we apply the criterion to obtain polynomial solutions to the generalized Hermite, Laguerre, Legendre and Chebyshev differential equations. As we shall show, the criterion presented here works whether or not the differential equation (2) has a set of orthogonal polynomial solutions, or a class of orthogonal polynomial solutions in the quasi-definite sense [10].

2. A criterion for polynomial solutions

The existence of polynomial solutions is characterized by the vanishing of δ_n. This is the principal theoretical result of this paper. We have:

Theorem 2. *(i) If the second-order differential equation (2) has a polynomial solution of degree n, then*

$$\lambda_n s_{n-1} - \lambda_{n-1} s_n \equiv \delta_n = 0. \tag{11}$$

Conversely (ii) if $\lambda_n\lambda_{n-1} \neq 0$, and $\delta_n = 0$, then the differential equation (2) has a polynomial solution whose degree is at most n.

Proof. (i) For the given differential equation (2), if y is a polynomial of degree at most n we have $y^{(n+1)} = y^{(n+2)} = 0$. Consequently we conclude from (8) that $\delta_n = 0$. (ii) Conversely, if $\delta_n = 0$ and $\lambda_n\lambda_{n-1} \neq 0$, then we have $s_{n-1}/\lambda_{n-1} = s_n/\lambda_n \equiv \alpha$, and, from theorem 1, we conclude that a solution is given by $y = \exp\left(-\int^x \alpha(t)\,\mathrm{d}t\right)$. Therefore, in particular, $y' = -\alpha y = -\frac{s_{n-1}}{\lambda_{n-1}} y$. Consequently, from $y^{(n+1)} = \lambda_{n-1}y' + s_{n-1}y$, we infer that $y^{(n+1)} = 0$, or, equivalently, that y is a polynomial of degree at most n. □

Theorem 2 gives us the condition under which the given differential equation has a polynomial solution. Theorem (1), in particular (9), provides a tool for the explicit computation of these polynomials. In the next section, we apply these results to a variety of classes of differential equations: in each case we provide the explicit condition which yields polynomial solutions.

3. Some differential equations with polynomial solutions

In this section, we apply theorem 2 to the classical differential equations of mathematical physics. First, we give an alternative proof to Bochner's results (1), using the criterion developed in theorem 2.

Theorem 3. *The second-order differential equation (1) has a polynomial solution of degree n if*

$$\mu_n = n(d + (n-1)a), \quad n = 0, 1, 2, \ldots. \tag{12}$$

The corresponding polynomial solutions are

$$\begin{aligned}
y_0 &= 1\\
y_1 &= dx + e\\
y_2 &= (d+a)(d+2a)x^2 + 2(b+e)(d+a)x + e(b+e) + c(d+2a)\\
y_3 &= (d+2a)(d+3a)(d+4a)x^3 + 3(d+2a)(d+3a)(e+2b)x^2\\
&\quad + 3(d+2a)(b(3e+2b) + c(4a+d) + e^2)x\\
&\quad + 4dbc + e^3 + 3dec + 10aec + 2eb^2 + 3e^2b\\
\cdots &= \cdots
\end{aligned}$$

Proof. By means of theorem 2, we find for $\lambda_0 = -\frac{dx+e}{ax^2+bx+c}$ and $s_0 = \frac{\mu}{ax^2+bx+c}$ that the termination condition $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1}s_n = 0$ yields

$$\delta_n = -\frac{1}{(ax^2+bx+c)^{n+1}}\prod_{k=0}^{n}(k(d+(k-1)a) - \mu_k), \quad n = 1, 2, \ldots \tag{13}$$

Table 1. Application of AIM to classical differential equations. For each differential equation which gives the condition under which it has polynomial solutions.

DE	λ_0	s_0	δ_n	$\delta_n = 0, n = 0, 1, \ldots$
Cauchy–Euler[1]	$\frac{\alpha(x-b)}{(x-a)^2}$	$\frac{\beta}{(x-a)^2}$	$\frac{(-1)^{n+1}}{(a-x)^{2n+2}}\prod_{i=1}^{n}(\beta+i(1-i+\alpha))$	$\beta = n(n-1-\alpha)$
Hermite[2a]	$2x$	$-2k$	$2^{n+1}\prod_{i=1}^{n}(k-i)$	$k = n$
Hermite[2b]	$ax+b$	c	$(-1)^{n+1}\prod_{i=0}^{n}(c+ia)$	$c = -na$
Laguerre	$(1-\frac{1}{x})$	$\frac{a}{x}$	$\frac{(-1)^{n+1}}{x^{n+1}}\prod_{i=0}^{n}(i+a)$	$a = -n$
Confluent[3]	$(b-\frac{c}{x})$	$\frac{a}{x}$	$\frac{(-1)^{n+1}}{x^{n+1}}\prod_{i=0}^{n}(ib+a)$	$a = -nb$
Hypergeometric	$\frac{(a+b+1)x-c}{x(1-x)}$	$\frac{ab}{x(1-x)}$	$\frac{1}{x^{n+1}(x-1)^{n+1}}\prod_{i=0}^{n}(a+i)(b+i)$	$a = -n$ or $b = -n$
Legendre	$\frac{2x}{1-x^2}$	$\frac{m(m+1)}{x^2-1}$	$\frac{(-1)^n}{(x^2-1)^{n+1}}\prod_{i=0}^{n}(m^2-i^2)$	$m = n$
Jacobi	$\frac{(\alpha+\beta+2)x+\beta+\alpha}{1-x^2}$	$\frac{-\gamma}{1-x^2}$	$\prod_{i=0}^{n}(i(i+1+\alpha+\beta)-\gamma)$	$\gamma = n(n+\alpha+\beta+1)$
Chebyshev[4a]	$\frac{x}{1-x^2}$	$\frac{-m}{1-x^2}$	$\frac{(-1)^{n+1}}{(x^2-1)^{n+1}}\prod_{i=0}^{n}(m-i^2)$	$m = n^2$
Chebyshev[4b]	$\frac{3x}{1-x^2}$	$\frac{-m}{1-x^2}$	$\frac{-1}{(x^2-1)^{n+1}}\prod_{i=0}^{n}(i((i+2)-m)$	$m = n(n+2)$
Gegenbauer	$\frac{(1+2k)x}{(1-x^2)}$	$\frac{-\lambda}{(1-x^2)}$	$\frac{-1}{(x^2-1)^{n+1}}\prod_{i=0}^{n}(i(i+2k)-\lambda)$	$\lambda = n(n+2k)$
Hyperspherical	$\frac{2(1+k)x}{(1-x^2)}$	$\frac{-\lambda}{(1-x^2)}$	$\frac{-1}{(x^2-1)^{n+1}}\prod_{i=0}^{n}(i(i+1+2k)-\lambda)$	$\lambda = n(n+1+2k)$
Bessel[5a]	$\frac{-2(x+1)}{x^2}$	$\frac{\gamma}{x^2}$	$\frac{(-1)^{n+1}}{x^{2n+2}}\prod_{i=0}^{n}(\gamma-i(i+1))$	$\gamma = n(n+1)$
Generalized Bessel[5b]	$\frac{-(ax+b)}{x^2}$	$\frac{\gamma}{x^2}$	$\frac{(-1)^{n+1}}{x^{2n+2}}\prod_{i=0}^{n}(\gamma-i(i-1+a))$	$\gamma = n(n+a-1)$

which yields for $\delta_n = 0$ that $\mu_n = n(d + (n-1)a)$ as required. For $n = 0, 1, 2, \ldots$ i.e. $\mu_0 = 0, \mu_1 = d, \mu_2 = 2(d+a), \ldots$, we obtian by

$$y_n = \exp\left(-\int^x \frac{s_n(t)}{\lambda_n(t)}\,\mathrm{d}t\right), \qquad n = 0, 1, 2, \ldots, \tag{14}$$

the polynomial solutions just mentioned. □

In table 1, we summarize the well-known differential equations which have polynomial solutions (as eigenfunctions). In each case, we give the explicit criterion, $\delta_n = 0$, of theorem 2.

3.1. Some remarks on table 1

1. This differential equation is a generalization of the Cauchy–Euler linear equation

$$x^2y'' + \alpha xy' + \beta y = 0. \tag{15}$$

It is possible, however, to apply AIM to the differential equation (15). The termination condition yields in this case

$$\delta_n = \frac{(-1)^{n+1}}{x^{2n+2}}\prod_{i=1}^{n}(\beta + i(1-i+\alpha)) = 0 \quad \text{or} \quad \beta = n(n-1-\alpha) \tag{16}$$

while the corresponding polynomials, as given by (14), are $y_0 = 1, y_1 = x, y_2 = x^2, \ldots, y_n = x^n$. It is clear that these polynomials cannot form an orthogonal-polynomial sequence [10].

2b. This differential equation can be regarded as a generalization of the well-known Hermite differential equation^{2a}. It is an elementary example of differential equation with non-rational coefficients (i.e. with s_0 and λ_0 non-rational) which has nonconstant polynomial solutions for $c \neq 0$.

3. This is known as the confluent hypergeometric differential equation. It is also known as Kummer's differential equation or Pochhammer–Barnes equation [19].

4a,b. This differential equation is known as Chebyshev's differential equation of the first kind and Chebyshev's differential equation of the second kind, respectively. It is interesting to note that these differential equations are special cases of

$$(1-x^2)y'' - axy' + \mu y = 0. \tag{17}$$

If we apply AIM directly to (17), we have by means of the termination condition (11) that

$$\delta_n = -\frac{1}{(x^2-1)^{n+1}} \prod_{k=0}^{n} (i(i+a-1) - \mu_i)$$

thus, for $\delta_n = 0$, we must have $\mu_n = n(n+a-1)$. The corresponding polynomial solutions, for $n = 0, 1, 2, \ldots,$ are $y_0 = 1$, $y_1 = x$, $y_2 = (a+1)x^2 - 1$, $y_3 = (a+1)x^3 - 3x, \ldots,$ and in general

$$y_n = {}_2F_1\left(-n, n+a-1, \frac{a}{2}, \frac{1-x}{2}\right)$$

up to a constant. Here, ${}_2F_1$, Gauss' hypergeometric function, is defined by

$${}_2F_1(-n, b; c; x) = \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{(c)_k k!} x^k, \tag{18}$$

where the Pochhammer symbol $(a)_k$ defined by

$$(a)_0 = 1, \qquad (a)_k = a(a+1)(a+2)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

5a,b. The polynomial solutions of these differential equations were studied by Krall and Frink [20]. The corresponding (Bessel) polynomial solutions are orthogonal in the quasi-definite sense [10].

In table 2 we find the corresponding polynomial solutions for each differential equation mentioned in table 1. As an elementary application to quantum mechanics, we consider the Schrödinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \left(-\frac{A}{r} + \frac{\gamma(\gamma+1)}{r^2}\right)\psi = E\psi.$$

Writing $\psi(r) = r^{\gamma+1}\,\mathrm{e}^{-\alpha r} y(r)$, we easily find that $y(r)$ must satisfy, for $E = -\alpha^2$, the confluent hypergeometric differential equation

$$y''(r) = 2\left(\alpha - \frac{\gamma+1}{r}\right)y'(r) + \left(\frac{-A + 2\alpha(\gamma+1)}{r}\right)y(r).$$

The termination condition, mentioned in table 1, then yields $E = -\alpha^2 = -\frac{A^2}{4(n+\gamma+1)^2}$, the eigenvalues of Schrödinger's equation for the Kratzer potential. Furthermore, the corresponding (un-normalized) eigenfunctions are given, by means of table 2, as

$$\psi_n(r) = (-1)^n r^{\gamma+1} e^{-\sqrt{-E}r} (2\gamma+2)_n {}_1F_1(-n; 2\gamma+2; 2\sqrt{-E}r).$$

Table 2. The corresponding polynomial solutions for each differential equation mentioned in table 1.

DE	$y_n, n = 0, 1, 2, \ldots$
Cauchy–Euler	$y_0 = 1$ $y_1 = x - b$ $y_2 = (\alpha - 1)(\alpha - 2)x^2 + 2(\alpha - 1)(2a - \alpha b)x$ $+\alpha^2 b^2 - a(2b + a)\alpha + 2a^2$ $\ldots$
Hermite	$y_0(x) = 1$ $y_1(x) = x$ $y_2(x) = 2x^2 - 1$ $\ldots$ $y_{2n}(x) = (-1)^n 2^n (1/2)_n {}_1F_1(-n; 1/2; x^2)$, $y_{2n+1}(x) = (-1)^n 2^n (3/2)_n x {}_1F_1(-n; 3/2; x^2)$
Hermite	$y_0(x) = 1$ $y_1(x) = ax + b$ $y_2(x) = (ax + b)^2 - a$ $\ldots$ $y_{2n}(x) = (-1)^n (2a)^n (1/2)_n {}_1F_1(-n; a/2; (ax + b)^2/2)$, $y_{2n+1}(x) = (-1)^n (2a)^n (3/2)_n (ax + b) {}_1F_1(-n; 3a/2; (ax + b)^2/2)$.
Laguerre	$y_0 = 1$ $y_1 = x - 1$ $y_2 = x^2 - 4x + 2$ $\ldots$ $y_n = (-1)^n n! {}_1F_1(-n, 1, x)$
Confluent	$y_0 = 1$ $y_1 = bx - c$ $y_2 = (1 + c)c - 2b(1 + c)x + b^2 x^2$ $\ldots$ $y_n = (-1)^n (c)_n {}_1F_1(-n, c, bx)$
Hypergeometric	$y_0 = 1$ $y_1 = x + c$ $y_2 = 2x^2 + 4(c + 1)x + c(c + 1)$ $\ldots$ $y_n = (c)_n {}_2F_1(-n, -n; c, x)$
Legendre	$y_0 = 1$ $y_1 = x$ $y_2 = -1 + x^2$ $\ldots$ $y_n = {}_2F_1(-n, 1 + n; 1, (1 - x)/2)$.
Jacobi	$y_0 = 1$ $y_1 = (\alpha - \beta) + (2 + \alpha + \beta)x$ $y_2 = (3 + \alpha + \beta)(4 + \alpha + \beta)x^2 + 2(\alpha - \beta)(3 + \alpha + \beta)x - 4 - c - d + (c - d)^2$ $\ldots$ $y_n = (\alpha + 1)_n / n! {}_2F_1(-n, n + \alpha + \beta + 1; \alpha + 1; (1 - x)/2)$.
Chebyshev	$y_0 = 1$ $y_1 = x$ $y_2 = 2x^2 - 1$ $\ldots$ $y_n = {}_2F_1(-n, n, \frac{1}{2}, (1 - x)/2)$

Table 2. (Continued.)

DE	$y_n, n = 0, 1, 2, \ldots$
Chebyshev	$y_0 = 1$ $y_1 = x$ $y_2 = 4x^2 - 1$ $\ldots$ $y_n = (n+1)_2F_1(-n, n+2, \frac{3}{2}, (1-x)/2)$
Gegenbauer	$y_0 = 1$ $y_1 = x$ $y_2 = 2(k+1)x^2 - 1$ $\ldots$ $y_n = (2k)_{n\,2}F_1(-n, n+2k; k+1/2; (1-x)/2)$
Hyperspherical	$y_0 = 1$ $y_1 = x$ $y_2 = (2k+3)x^2 - 1$ $\ldots$ $y_n = (2k+1)_{n\,2}F_1(-n, n+2k+1; k+1; (1-x)/2)$
Bessel	$y_1(x) = 1 + x$
Polynomials	$y_2(x) = 1 + 3x + 3x^2$ $y_3(x) = 1 + 6x + 15x^2 + 15x^3$ $\ldots$ $y_n(x) = {}_2F_0(-n, n+1; -; -x/2)$
Generalized Bessel	$y_1(x) = ax + b$
Polynomials	$y_2(x) = (a+1)(a+2)x^2 + 2b(a+1)x + b^2$ $y_3(x) = (a+2)(a+3)(a+4)x^3 + 3b(a+2)(a+3)x^2 + 3b^2(2+a)x + b^3$ $\ldots$ $y_n(x) = b^n{}_2F_0(-n, n+a-1; -; -x/b)$

3.2. *The case of* $\lambda_0 = 0$

In the early development of the asymptotic iteration method [11], one gets the impression that the method is not applicable in the case of $\lambda_0 = 0$. This impression naturally arises because of the condition $\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}$, $n = 0, 1, 2, \ldots$. If $\lambda_0 = 0$, however, we may have using (5) and (6) that $\frac{y_{n+2}}{y_{n+1}} = \frac{s_n(\frac{\lambda_n}{s_n}y'+y)}{s_{n-1}(\frac{\lambda_{n-1}}{s_{n-1}}y'+y)}$ for which the corresponding asymptotic condition now reads

$$\frac{\lambda_n}{s_n} = \frac{\lambda_{n-1}}{s_{n-1}} \equiv \alpha, \qquad n = 1, 2, \ldots \tag{19}$$

This leads to the essentially equivalent termination condition (11)

$$\delta_n = \lambda_n s_{n-1} - \lambda_{n-1}s_n = 0, \qquad n = 1, 2, \ldots$$

A simple example which shows the use of AIM in the case of $\lambda_0 = 0$ is the differential equation $x^2y'' - 2y = 0$. Direct use of AIM implies that $\delta_2 = 0$ and a polynomial solution by means of (14) is $y = x^2$.

4. Application to generalized Hermite, Laguerre, Legendre and Chebyshev differential equations

Theorem 4. *For N a positive integer and $a, b \neq 0$, the second-order linear differential equation (known as the generalized Laguerre differential equation)*

$$u'' = \left(ax^N - \frac{b}{x}\right)u' - acx^{N-1}u, \tag{20}$$

has a polynomial solution if

$$c = n(N+1), \qquad n = 0, 1, 2, \ldots. \tag{21}$$

The corresponding polynomial solutions are

$$u_n(x) = (N+1)^n \left(\frac{b+N}{1+N}\right)_n {}_1F_1\left(-n; \frac{b+N}{1+N}; \frac{ax^{N+1}}{1+N}\right). \tag{22}$$

Proof. For $N = 1$, the termination condition (11) yields $c = 2n, n = 0, 1, 2, \ldots$ while (14) implies

$$u_n(x) = \begin{cases} 1, & \text{if } n = 0 \text{ (or } c = 0) \\ 1 + b - ax^2, & \text{if } n = 1 \text{ (or } c = 2) \\ 3 + 4b + b^2 - 2a(3+b)x^2 + a^2x^4, & \text{if } n = 2 \text{ (or } c = 4) \\ \ldots & \\ 2^n \left(\frac{b+1}{2}\right)_n {}_1F_1\left(-n; \frac{b+1}{2}; \frac{ax^2}{2}\right), & \text{for } n = 0, 1, 2, \ldots \text{ (or } c = 2n). \end{cases}$$

For $N = 2$, the termination condition (11) yields $c = 3n, n = 0, 1, 2, \ldots$ while (14) implies

$$u_n(x) = \begin{cases} 1, & \text{if } n = 0 \text{ (or } c = 0) \\ 2 + b - ax^3, & \text{if } n = 1 \text{ (or } c = 3) \\ 10 + 7b + b^2 - 2a(5+b)x^3 + a^2x^6, & \text{if } n = 2 \text{ (or } c = 6) \\ \ldots & \\ 3^n \left(\frac{b+2}{3}\right)_n {}_1F_1\left(-n; \frac{b+2}{3}; \frac{ax^3}{3}\right), & \text{for } n = 0, 1, 2, \ldots \text{ (or } c = 3n). \end{cases}$$

Similarly, for $N = 3$, the termination condition (11) yields $c = 4n, n = 0, 1, 2, \ldots$ while (14) implies

$$u_n(x) = \begin{cases} 1, & \text{if } n = 0 \text{ (or } c = 0) \\ 3 + b - ax^4, & \text{if } n = 1 \text{ (or } c = 4) \\ 21 + 10b + b^2 - 2a(7+b)x^4 + a^2x^8, & \text{if } n = 2 \text{ (or } c = 8) \\ \ldots & \\ 4^n \left(\frac{b+3}{4}\right)_n {}_1F_1\left(-n; \frac{b+3}{4}; \frac{ax^4}{4}\right), & \text{for } n = 0, 1, 2, \ldots \text{ (or } c = 4n). \end{cases}$$

Similar expressions can be obtained for $N = 4, 5, \ldots$. These results can be generalized by (22). □

Theorem 5. *If $b = 0$, the second-order linear differential equation (20), known as generalized Hermite differential equation, has a polynomial solution if*

$$c = n(N+1), \quad n = 0, 1, 2, \ldots \tag{23}$$

or

$$c = n(N+1)+1 \quad n = 0, 1, 2, \ldots \tag{24}$$

In the case of $c = n(N+1)$*, the polynomial solutions are (for* $n = 0, 1, 2, \ldots$*)*

$$u_n(x) = (-1)^n (N+1)^n \left(\frac{N}{1+N}\right)_n {}_1F_1\left(-n; \frac{N}{1+N}; \frac{ax^{N+1}}{1+N}\right). \tag{25}$$

In the case of $c = n(N+1)+1$*, the polynomial solutions are (for* $n = 0, 1, 2, \ldots$*)*

$$u_n(x) = (-1)^n (N+1)^n \left(\frac{2+N}{1+N}\right)_n x\, {}_1F_1\left(-n; \frac{2+N}{1+N}; \frac{ax^{N+1}}{1+N}\right). \tag{26}$$

Proof. Similarly to the proof of theorem 4, the conditions (23) and (24) follow directly by means of the termination condition (11), with $\lambda_0 = ax^N$ and $s_0 = -acx^{N-1}$. Equations (25) and (26) follow from (14). □

Theorem 6. *For N a positive integer, the differential equation*

$$u'' = \left(\frac{ax^N}{1-sx^{N+1}} - \frac{b}{x}\right)u' - \frac{wx^{N-1}}{1-sx^{N+1}}u, \tag{27}$$

has polynomial solutions

$$u_n(x) = \frac{(-1)^n}{(N+1)^{-n}}\left(\frac{N+b}{N+1}\right)_n {}_2F_1\left(-n, \frac{b-1}{N+1} + \frac{a}{(N+1)s} + n; \frac{b+N}{1+N}; sx^{N+1}\right) \tag{28}$$

if

$$w = n(N+1)(s(b-1+n(N+1))+a), \quad n = 0, 1, 2, \ldots \tag{29}$$

where ${}_2F_1$ *is Gauss's hypergeometric function (18). If* $b = 0, s = 1$*, then the cases of* $a = 2$ *and* $a = 1$ *correspond to differential equations known as the generalized Legendre and Chebyshev differential equations, respectively.*

Proof. Using AIM, condition (29) for polynomial solutions follows by means of the termination condition $\delta_n = 0$ in a similar fashion to the proof of theorem 4. Equation (28) then follows by means of (14) as generalization of the polynomial solutions for each of $n = 0, 1, 2, \ldots$ and $N = 1, 2, \ldots$. □

5. Conclusion

We have presented a simple criterion for the existence of polynomial solutions of second-order linear differential equations. Many of the classical differential equations that appear in mathematical physics can be analysed with this theory. Apart from its theoretical interest, the criterion can be used in a practical way to look for and to obtain polynomial solutions to eigenvalue problems of Schrödinger-type [11–18], and similarly for polynomial solutions of quasi-exact solvable models in quantum mechanics [21].

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